Auditorium Exercise Sheet 2

Differential Equations I for Students of Engineering Sciences

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General information on the DGL I course

- Lecturer: Professor Thomas Schmidt
- Lectures (English, weekly): Tue 16:45-18:15 Audimax II

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- Auditorium Exercise class (English, bi-weekly): Mo 09:45-11:15 H0.16
- Exercise groups (English, bi-weekly):
  - Mo 11:30-13:00 H0.01
  - Mo 16:00-17:30 N0009
  - Tue 08:00-09:30 O-007
- Exercises and Homework at: DGL I - Lecture material WiSe 2023/2024

Table of contents

Introduction to differential equations

- Previously: given an algebraic equation/system, look for solution(s) among a certain space of numbers/vectors.
  Now: given a differential equation (ODE), look for solution(s) in a space of functions.
- A (real, scalar) ODE is an equation in which a function $y = y(t)$ and its derivative(s) $y', y'', \dots, y^{(m)}$ (up to order $m \in \mathbb{N}$) are related:

$$F(t, y, y', y'', \dots, y^{(m)}) = 0 \rightarrow m\text{-th order ODE in implicit form}$$

$$y^{(m)} = f(t, y, y', y'', \dots, y^{(m-1)}) \rightarrow m\text{-th order ODE in explicit form}$$

  where $y : I \rightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ domain of definition.
- A given function $\overline{y}$ defined on I is a **solution** of $F(t, y, y', y'', \dots, y^{(m)}) = 0$ if

$$F(t, \overline{y}(t), \overline{y}'(t), \overline{y}''(t), \dots, \overline{y}^{(m)}(t)) = 0 \quad \text{for all } t \in I.$$

Introduction to differential equations

- Notice that in case an ODE admits a solution $\overline{y}$ on I, then we cannot expect $\overline{y}$ to be unique!
- The collection of all the possible solution of an ODE is called **general solution** of the ODE.

- Example 1: $y' = 4y$, where $y = y(t)$ defined on $I = \mathbb{R}$.

$$y \equiv 0 \text{ is a (trivial) solution}$$

$$y(t) = e^{4t} \text{ solves the ODE, but also } y(t) = 7e^{4t}!$$

The general solution of $y' = 4y$ is given by $y(t) = \mathbf{C}e^{4t}, \; C \in \mathbb{R}$.

CHECK: $y'(t) = (Ce^{4t})' = 4Ce^{4t} = 4y(t)$ ✓

In order to get a unique solution, we need to impose some restrictions to the ODE: initial and/or boundary conditions.

- Example of **initial value problem** (IVP):

$$\begin{cases} y'' = f(t, y, y') & \leftarrow \text{ ODE of order } m = 2 \\ y(t_0) = y_0 & \\ y'(t_0) = z_0 & \leftarrow \text{ need 2 conditions (on } y \text{ and } y') \end{cases}$$

  with $t_0 \in I$, $y_0, z_0 \in \mathbb{R}$.

- Example of **boundary value problem** (BVP):

$$\begin{cases} y'' = f(t, y, y') & \leftarrow \text{ ODE of order } m = 2 \\ y(a) = y_a & \\ y(b) = y_b & \leftarrow \text{ need 2 boundary values}^* \end{cases}$$

  where $I = [a, b]$ and $y_a, y_b \in \mathbb{R}$.

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* In general, boundary condition of the form $r_i(y(a), y(b)) = 0$, for $i \in \{1, \ldots, m\}$.

Example of initial value problem

- Solve the ODE in Example 1 with initial datum $y(2) = 5$, i.e. solve IVP given by the system

$$\begin{cases} y' = 4y \\ y(2) = 5. \end{cases}$$

Consider the general solution $y(t) = Ce^{4t}, \; C \in \mathbb{R}$ of the ODE and find the appropriate C by substituting the initial condition:

$$5 = y(2) = Ce^8 \implies C = 5e^{-8},$$

therefore the (unique!) solution of the IVP is given by $y = 5e^{4(t-2)}$.

Autonomous differential equations

An m-th order ODE in $y = y(t)$ is called **autonomous** if the independent variable t does NOT appear in its expression, i.e. the equation is of the form

$$y^{(m)} = f(y, y', y'', \ldots, y^{(m-1)})$$

(instead of $y^{(m)} = f(\mathbf{t}, y, y', y'', \ldots, y^{(m-1)})$!)

- Examples of autonomous differential equations:

$$y' = 4y; \qquad 5y''' - y' = y^2 - 1; \qquad 9y''/y - y + y^3 = 7$$

- Examples of NON autonomous differential equations:

$$y' = 4y + 2e^t; \qquad 5\cos(t)y''' - y' = y^2; \qquad 9y''/ty - y + y^3 = t^7$$

Linear differential equations

An m-th order ODE in $y = y(t)$ is **linear** if the ODE is a linear combination of the derivatives $y, y', \ldots, y^{(m)}$, i.e. it is of the form

$$A_m(t)y^{(m)} + A_{m-1}(t)y^{(m-1)} + \cdots + A_2(t)y'' + A_1(t)y' + A_0(t)y = b(t) \quad (1)$$

with coefficients given by the functions $A_0, A_1, \ldots, A_m : I \to \mathbb{R}$ and inhomogeneity term $b : I \to \mathbb{R}$.

Linear differential equations

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with coefficients given by the functions $A_0, A_1, \dots, A_m : I \to \mathbb{R}$ and inhomogeneity term $b : I \to \mathbb{R}$.

- If the coefficients $A_0, A_1, \dots, A_m$ are constant in I, we say that (1) is a linear ODE with **constant coefficients**.
- In case $b \equiv 0$ in I, the equation (1) is called **homogeneous**, otherwise it is **inhomogeneous**.
- For a linear ODE (1), the **corresponding homogeneous** ODE is the equation

$$A_m(t)y^{(m)} + A_{m-1}(t)y^{(m-1)} + \dots + A_2(t)y'' + A_1(t)y' + A_0(t)y = 0. \quad (2)$$

Examples of linear/non-linear ODEs

- $y' = 4y$ (lin. hom. const. coeff.);
- $y' = 4y + 2e^t$ (lin. inhom. const. coeff.);
- $9y''/y - y + y^3 = 7$ (non-lin. inhom.) $\rightsquigarrow$ corr. hom.: $9y''/y - y + y^3 = 0$;
- $5\cos^4(t)y''' - y' + t^2 = 0$ (lin. inhom., non-const. coeff.) $\rightsquigarrow$ corr. hom.: $5\cos^4(t)y''' - y' = 0$;
- $y''' - 3y' = y^2$ (non lin. hom.);

Resolution of first order linear ODEs

A linear ODE of order $m = 1$ is of the kind

$$y' = a(t)y + b(t).$$

For $a, b : I \to \mathbb{R}$ continuous, there is an explicit formula for determining its general solution.

Resolution of **HOMOGENEOUS** linear ODEs of order 1

Given a linear, homogeneous, 1^{st} order ODE

$$y' = a(t)y, \tag{3}$$

with $a : I \to \mathbb{R}$ continuous, let $A(t) := \int a(t)\mathrm{d}t$. Then the general solution of (3) is determined by

$$y(t) = Ce^{A(t)}, \qquad C \in \mathbb{R}.$$

Notice that we find infinite solutions depending on the real parameter C!

Resolution of first order linear ODEs

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$$y' = a(t)y + b(t).$$

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Resolution of INHOMOGENEOUS linear ODEs of order 1

Given a linear 1st order ODE

$$y' = a(t)y + b(t), \tag{4}$$

with $a, b : I \to \mathbb{R}$ continuous, let $A(t) := \int a(t)\mathrm{d}t$ and $B^*(t) := \int b(t)e^{-A(t)}\mathrm{d}t$. Then the general solution of (3) is determined by

$$y(t) = e^{A(t)}(B^*(t) + C), \qquad C \in \mathbb{R}.$$

Notice that we find infinite solutions depending on the real parameter C!

Example of resolution of HOMOGENEOUS linear ODEs of order 1

Example 2. Find the general solution $y = y(t)$, $t \in \mathbb{R}$ of

$$y' - 6t^2 y = 0. \tag{5}$$

It is a first order, linear ODE with $a(t) = 6t^2$ continuous and $b(t) \equiv 0$.
Employ the formula for homogeneous ODEs: $y(t) = Ce^{A(t)}$, $C \in \mathbb{R}$.
Compute $A(t) := \int a(t)\mathrm{d}t = \int 6t^2 = 2t^3$ (+const., set const. = 0), thus the general solution of (5) is given by:

$$y(t) = Ce^{2t^3}, \qquad C \in \mathbb{R}.$$

If time available: CHECK that Ce^{2t^3} is solution ...

Example of resolution of INHOMOGENEOUS linear ODEs of order 1

Example 3. Find the general solution $y = y(t)$, $t \in \mathbb{R}$ of

$$y' - 6t^2 y = t^2. \tag{6}$$

It is a first order, linear ODE with $a(t) = 6t^2$ and $b(t) = t^2$. Employ the formula for inhomogeneous ODEs: $y(t) = e^{A(t)}(B^*(t) + C)$, $C \in \mathbb{R}$. Compute $A(t) := \int a(t)\mathrm{d}t = \int 6t^2 = 2t^3$ (+const., set const. = 0) and $B^*(t) := \int b(t)e^{-A(t)}\mathrm{d}t = \int t^2 e^{-2t^3}\mathrm{d}t = -e^{-2t^3}/6$. The general solution of (6) is given by:

$$y(t) = e^{2t^3}(-e^{-2t^3}/6 + C) = Ce^{2t^3} - 1/6, \qquad C \in \mathbb{R}.$$

If time available: CHECK that $Ce^{2t^3} - 1/6$ is solution ...

Example of resolution of INHOMOGENEOUS linear ODEs of order 1

Example 3. Find the general solution $y = y(t)$, $t \in \mathbb{R}$ of

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It is a first order, linear ODE with $a(t) = 6t^2$ and $b(t) = t^2$. Employ the formula for inhomogeneous ODEs: $y(t) = e^{A(t)}(B^*(t) + C)$, $C \in \mathbb{R}$. Compute $A(t) := \int a(t)\mathrm{d}t = \int 6t^2 = 2t^3$ (+const., set const. = 0) and $B^*(t) := \int b(t)e^{-A(t)}\mathrm{d}t = \int t^2 e^{-2t^3}\mathrm{d}t = -e^{-2t^3}/6$. The general solution of (6) is given by:

$$y(t) = e^{2t^3}(-e^{-2t^3}/6 + C) = Ce^{2t^3} - 1/6, \qquad C \in \mathbb{R}.$$

NOTE: The general integral of (6) is given by the difference of $y_{\mathrm{hom}} := Ce^{2t^3}$ gen. int. of (5), and $y_{\mathrm{p}} := -1/6$ particular sol. of (6).

Exercises

Exercise 1. For each of the following differential equation, determine:

- the order;
- if the ODE is autonomous or not;
- if the ODE is linear or not;
- if it is homogeneous or not;

In case of linearity, determine:

- if it has continuous and/or constant coefficients;

In case of inhomogeneity, write the corresponding homogeneous ODE.

(i) $y' = 6t^2 y;$ (ii) $y'' = 4t^3\sqrt{t}, \ t \geq 0;$

(iii) $y' + \frac{2t}{y^{(4)}}(1 + 2t^2) = 0;$ (iv) $y' = t(y + e^4);$

(v) $y''' = y^2 - 1;$ (vi) $\dot{x} = (x^2 + t^2)/tx, \ t \neq 0;$

(vii) $y' + \cos^5(t)y - ty^3 = 0;$ $(viii)$ $y'' = y + 2t.$

Exercises

- Exercise 2. Determine the general solutions of the following linear differential equations (unless specified, consider $I = \mathbb{R}$).

(i) $y' - 5y + 10 = t$

(ii) $\dot{x} = \cos(5t)(x+1)$;

(iii) $y' + 2ty = e^{t-t^2}$;

(iv) $y' + 4t^3 y - \sin(t)e^{-t^4} = 0$;

(v) $y'/2 - y = 2t^2$

(vi) $y' = \dfrac{1}{t^2 - 1}, -1 < t < 1$;

(vii) $(y')^2 = y' \sin(3t)$.

- Exercise 3. For each ODE of Exercise 2, determine the (unique) solution knowing that the function takes value -1 at time $t = 0$.

Exercises

Exercise 4. Consider the differential equation

$$y'' + 9y - 3t = 0 \tag{7}$$

and the functions $y_1(t) := \cos(3t)$, $y_2(t) := \sin(3t)$, $y_3(t) := t/3$.

- Classify the equation (7).
- Verify that the functions $y_1 + y_3$, $y_2 + y_3$ and y_3 are solutions of (7).
- Write the associated homogeneous equation of (7).
- Verify that any linear combination of y_1, y_2 is a solution of the homogeneous equation of (7).
- Can you find a linear combination of y_1 and y_3 which does NOT solve (7)? Why?

Appendix

Table of most common integrals

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \frac{1}{x}\,dx = \ln|x|$
3. $\int e^x\,dx = e^x$
4. $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$
8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$
10. $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\int \csc x\,dx = \ln|\csc x - \cot x|$
13. $\int \tan x\,dx = \ln|\sec x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sinh x\,dx = \cosh x$
16. $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

*19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Appendix

Some trigonometric identities

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta - \phi) - \cos(\theta + \phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta - \phi) + \cos(\theta + \phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta + \phi) + \sin(\theta - \phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta + \phi) - \sin(\theta - \phi)]}{2}$$

DGL I - AUDITORIUM EXERCISE CLASS 2

Exercise 2

$$\left[y'(t) = a(t)\cdot y(t) + b(t) \right]$$

1st order linear ODE with continuous coeff.

general solution:

$$y(t) = e^{A(t)}\left[B^*(t) + C \right]$$

(i) $y' - 5y + 10 = t$, $t \in \mathbb{R} = I$

$$\left[y' = 5y + (t-10) \right]$$

$a(t) = 5$

$b(t) = t - 10$

$$A(t) = \int a(t)\,dt = \int 5\,dt = 5t + \cancel{const}_{=0}$$

$$B^*(t) = \int b(t)\cdot e^{-A(t)}\,dt = \int (t-10)e^{-5t}\,dt =$$

$$= \underline{\int t e^{-5t}\,dt} - 10\int e^{-5t}\,dt =$$

$$= -\frac{te^{-5t}}{5} + \int \frac{e^{-5t}}{5}\,dt + 2e^{-5t} =$$

$$= -\frac{t}{5}e^{-5t} - \frac{e^{-5t}}{25} + 2e^{-5t} =$$

$$\ldots = e^{-5t}\cdot\frac{49-5t}{25}.$$

$\int$ polyn. exp: polyn. $\to f$, exp $\to g'$

$g'(t) = e^{-5t}$ $\quad g(t) = -\frac{e^{-5t}}{5}$

$f(t) = t$ $\quad f'(t) = 1$

INT. BY PARTS

$$\int f(t)\cdot g'(t)\,dt = f(t)\cdot g(t) - \int f'(t)\cdot g(t)\,dt$$

$$\rightsquigarrow y(t) = e^{5t}\left(e^{-5t}\cdot\frac{49-5t}{25} + C\right) = \frac{49-5t}{25} + Ce^{5t}, \quad C \in \mathbb{R}$$

$\hookrightarrow$ general sol. of the ODE

+ Exercise 3(i)

Solve (IVP) $\begin{cases} y' - 5y + 10 = t \leftarrow \\ y(0) = -1 \end{cases}$ ($t=0$)

The sol. of the (IVP) is:

$$y(t) = \frac{49-5t}{25} - \frac{74}{25}e^{5t}$$

$$-1 = y(0) = \frac{49 - 5\cdot 0}{25} + C\cdot e^{5\cdot 0} = \frac{49}{25} + C \Rightarrow C = \frac{-25-49}{25} = \frac{-74}{25}$$

Exercise 3(vi)

(IVP) $\begin{cases} y' = \dfrac{1}{t^2-1} \\ y(0) = -1 \end{cases}$

$-1 < t < 1$ ($0 \in I$ ✓)

$\left[y' = a(t)\cdot y + b(t) \right.$ with $a(t) = 0$, $b(t) = \frac{1}{t^2-1}$

General sol. of the ODE

$$y(t) = \underbrace{e^{A(t)}}_{=1}\left(B^*(t) + C\right) = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$\hookrightarrow -1 = y(0) = \frac{1}{2}\ln\left|\frac{-1}{1}\right| + C = C \Rightarrow C = -1$

$$A(t) = \int 0\,dt = 0 + \cancel{const}$$

$$B^*(t) = \int b(t)\cdot \underbrace{e^{-A(t)}}_{=1}\,dt = \int \frac{1}{t^2-1}\,dt = \int \frac{1}{2(t-1)}\,dt - \int \frac{1}{2(t+1)}\,dt = \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + \cancel{const}_{0} =$$

$(\ln|x|)' = \frac{1}{x}$

$$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|$$

PARTIAL FRACTION DECOMPOSITION

$$\left[\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{At + A + Bt - B}{(\;)} \rightsquigarrow \begin{cases} (A+B)t = 0 \\ A - B = 1 \end{cases} \Rightarrow \begin{cases} A = -B = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases} \right]$$